Probability Distributions

(Slides used with permission)

Author: Kristin L. Sainani, PhD
Associate Professor with Health Research and Policy at Stanford University
Webpage: https://web.stanford.edu/~kcobb/

1

Random Variable

- A random variable *x* takes on a defined set of values with different probabilities.
  - For example, if you roll a die, the outcome is random (not fixed) and there are 6 possible outcomes, each of which occur with probability one-sixth.
  - For example, if you poll people about their voting preferences, the percentage of the sample that responds "Yes on Proposition 100" is also a random variable (the percentage will be slightly differently every time you poll).

- Roughly, probability is how frequently we expect different outcomes to occur if we repeat the experiment over and over ("frequentist" view)

2

Random variables can be discrete or continuous

- **Discrete** random variables have a countable number of outcomes
  - Examples: Dead/alive, treatment/placebo, dice, counts, etc.
- **Continuous** random variables have an infinite continuum of possible values.
  - Examples: blood pressure, weight, the speed of a car, the real numbers from 1 to 6.

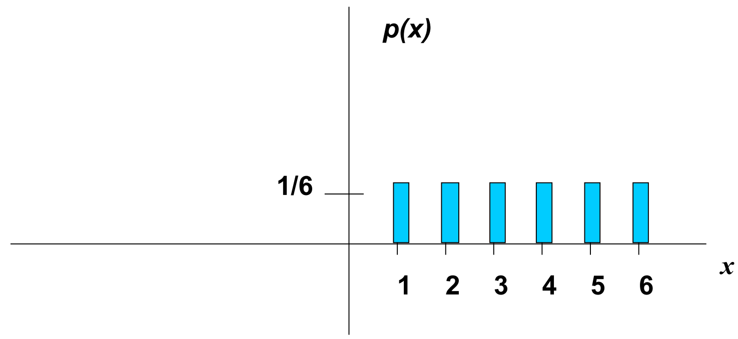
3

Probability functions

- A probability function maps the possible values of *x* against their respective probabilities of occurrence, *p(x)*
- *p(x)* is a number from 0 to 1.0.
- The area under a probability function is always 1.

4

Discrete example: roll of a die



$$\sum_{\text{all } x} P(x) = 1$$

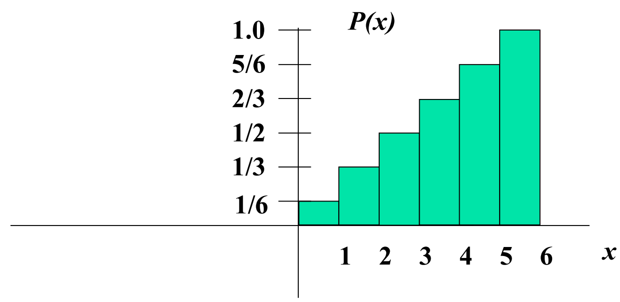
5

Probability mass function (pmf)

x	*p(x)*
1	*p(x=1)*=1/6
2	*p(x=2)*=1/6
3	*p(x=3)*=1/6
4	*p(x=4)*=1/6
5	*p(x=5)*=1/6
6	*p(x=6)*=1/6
	1.0

6

Cumulative distribution function (CDF)



7

Cumulative distribution function

x	P(x≤A)
1	P(x≤1)=1/6
2	P(x≤2)=2/6
3	P(x≤3)=3/6
4	P(x≤4)=4/6
5	P(x≤5)=5/6
6	P(x≤6)=6/6

8

Examples

1. What's the probability that you roll a 3 or less?

$P(x \leq 3) = 1/2$

2. What's the probability that you roll a 5 or higher?

$P(x \geq 5) = 1 - P(x \leq 4) = 1\text{-}2/3 = 1/3$

9

Practice Problem

Which of the following are probability functions?

a. $f(x)=.25$ for x=9,10,11,12

b. $f(x)= (3\text{-}x)/2$ for x=1,2,3,4

c. $f(x)= (x^2+x+1)/25$ for x=0,1,2,3

10

Answer (a)

a. *f(x)=.25* for x=9,10,11,12

x	*f(x)*
9	.25
10	.25
11	.25
12	.25
	1.0

Yes, probability function!

11

Answer (b)

b. *f(x)= (3-x)/2* for x=1,2,3,4

x	*f(x)*
1	(3-1)/2=1.0
2	(3-2)/2=.5
3	(3-3)/2=0
4	(3-4)/2=-.5

Though this sums to 1, you can't have a negative probability; therefore, it's not a probability function.

12

Answer (c)

c. $f(x) = (x^2+x+1)/25$ for x=0,1,2,3

x	f(x)
0	1/25
1	3/25
2	7/25
3	13/25

24/25

Doesn't sum to 1. Thus, it's not a probability function.

13

Practice Problem:

- The number of ships to arrive at a harbor on any given day is a random variable represented by *x*. The probability distribution for *x* is:

x	10	11	12	13	14
P(x)	.4	.2	.2	.1	.1

Find the probability that on a given day:

a. exactly 14 ships arrive $p(x=14) = .1$

b. At least 12 ships arrive $p(x \geq 12) = (.2 + .1 + .1) = .4$

c. At most 11 ships arrive $p(x \leq 11) = (.4 + .2) = .6$

14

Practice Problem:

You are lecturing to a group of 1000 students. You ask them to each randomly pick an integer between 1 and 10. Assuming, their picks are truly random:

- What's your best guess for how many students picked the number 9?
  - Since $p(x=9) = 1/10$, we'd expect about 1/10th of the 1000 students to pick 9. 100 students.
- What percentage of the students would you expect picked a number less than or equal to 6?
  - Since $p(x \leq 6) = 1/10 + 1/10 + 1/10 + 1/10 + 1/10 + 1/10 = .6$ 60%

15

Important discrete distributions in epidemiology…

- Binomial
  - Yes/no outcomes (dead/alive, treated/untreated, smoker/non-smoker, sick/well, etc.)
- Poisson
  - Counts (e.g., how many cases of disease in a given area)

16

Continuous case

- The probability function that accompanies a continuous random variable is a continuous mathematical function that integrates to 1.
- The probabilities associated with continuous functions are just areas under the curve (integrals!).
- Probabilities are given for a range of values, rather than a particular value (e.g., the probability of getting a math SAT score between 700 and 800 is 2%).

17

Continuous case

- For example, recall the negative exponential function (in probability, this is called an "exponential distribution"): $f(x) = e^{-x}$

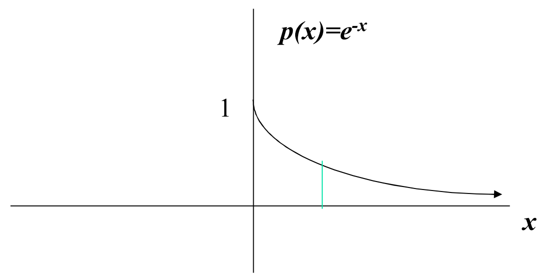
  - This function integrates to 1:

$$\int_0^{+\infty} e^{-x} = -e^{-x} \Big|_0^{+\infty} = 0 + 1 = 1$$

e is approximately equal to 2.71828.

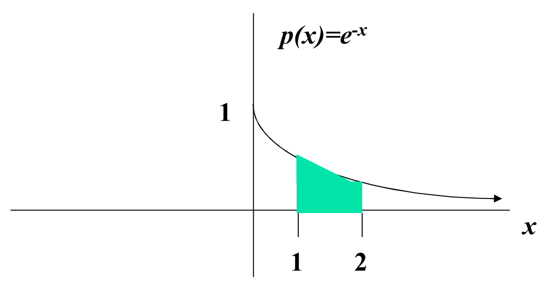
18

Continuous case: "probability density function" (pdf)



The probability that *x* is any exact particular value (such as 1.9976) is 0; we can only assign probabilities to possible ranges of x.

For example, the probability of *x* falling within 1 to 2:



$$P(1 \le x \le 2) = \int_1^2 e^{-x} = -e^{-x} \Big|_1^2 = -e^{-2} - -e^{-1} = -.135 + .368 = .23$$

Cumulative distribution function

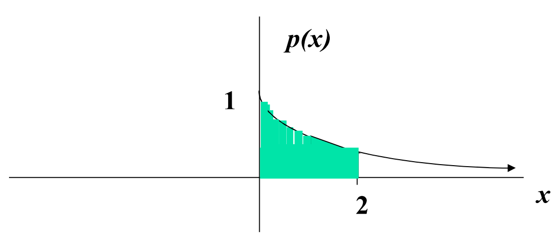
As in the discrete case, we can specify the "cumulative distribution function" (CDF):

The CDF here = P(x≤A)=

$$\int_0^A e^{-x} = -e^{-x} \Big|_0^A = -e^{-A} - -e^{0} = -e^{-A} + 1 = 1 - e^{-A}$$

21

Example



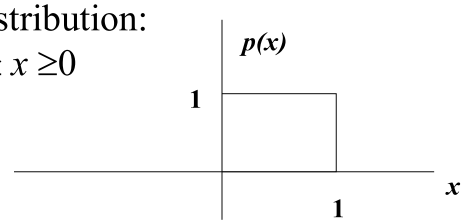
$$\text{P}(\text{x} \le 2) = 1 - e^{-2} = 1 - .135 = .865$$

22

Example 2: Uniform distribution

The uniform distribution: all values are equally likely

The uniform distribution:
f(x)= 1 , for 1≥ *x* ≥0



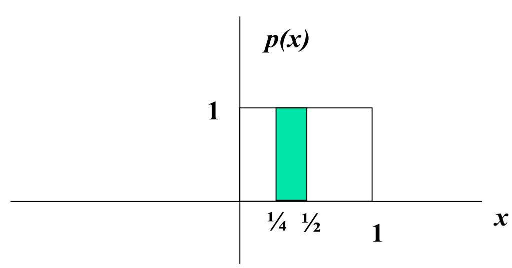
We can see it's a probability distribution because it integrates to 1 (the area under the curve is 1):

$$\int_0^1 1 = x \Big|_0^1 = 1 - 0 = 1$$

23

Example: Uniform distribution

What's the probability that *x* is between ¼ and ½?



P(½ ≥*x*≥ ¼)= ¼

24

Practice Problem

4. Suppose that survival drops off rapidly in the year following diagnosis of a certain type of advanced cancer. Suppose that the length of survival (or time-to-death) is a random variable that approximately follows an exponential distribution with parameter 2 (makes it a steeper drop off):

$$\text{probability function}: p(x=T)=2e^{-2T}$$

$$[note: \int_0^{+\infty} 2e^{-2x} = -e^{-2x} \Big|_0^{+\infty} = 0+1=1]$$

What's the probability that a person who is diagnosed with this illness survives a year?

25

Answer

The probability of dying within 1 year can be calculated using the cumulative distribution function:

Cumulative distribution function is:

$$P(x \le T) = -e^{-2x} \Big|_0^{T} = 1-e^{-2(T)}$$

The chance of surviving past 1 year is: *P(x≥1) = 1 – P(x≤1)*

$$1-(1-e^{-2(1)}) = .135$$

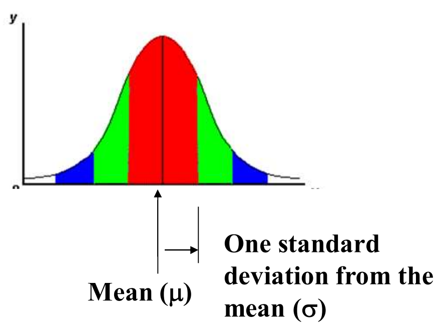
26

Expected Value and Variance

- All probability distributions are characterized by an expected value and a variance (standard deviation squared).

27

For example, bell-curve (normal) distribution:



28

Expected value, or mean

- If we understand the underlying probability function of a certain phenomenon, then we can make informed decisions based on how we expect *x* to behave on-average over the long-run…(so called "frequentist" theory of probability).

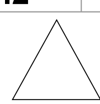
- Expected value is just the weighted average or mean (μ) of random variable *x*. Imagine placing the masses *p*(*x*) at the points *X* on a beam; the balance point of the beam is the expected value of *x*.

29

Example: expected value

- Recall the following probability distribution of ship arrivals:

x	**10**	**11**	**12**	**13**	**14**
P(x)	**.4**	**.2**	**.2**	**.1**	**.1**



$$\sum_{i=1}^{5} x_i\, p(x) = 10(.4) + 11(.2) + 12(.2) + 13(.1) + 14(.1) = 11.3$$

30

Expected value, formally

Discrete case:

$$E(X) = \sum_{\text{all } x} x_i p(x_i)$$

General Formula:

Outcome	x_1	x_2	x_3	x_4	...	x_n
Probability	p_1	p_2	p_3	p_4	...	p_n

Expected Value = $p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4 + \dots + p_nx_n$

Example: A single fair six-sided die is rolled.

Outcome	1	2	3	4	5	6
Probability	1/6	1/6	1/6	1/6	1/6	1/6

Expected Value = $1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$

Continuous case:

$$E(X) = \int_{\text{all } x} x_i p(x_i) dx$$

The symbol dx, called the differential of the variable x, indicates that the variable of integration is x.

31

Empirical Mean is a special case of Expected Value…

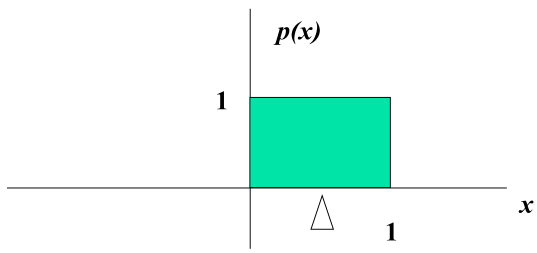
Sample mean, for a sample of n subjects: =

$$\overline{X} = \frac{\sum_{i=1}^{n} x_i}{n} = \sum_{i=1}^{n} x_i \left(\frac{1}{n}\right)$$

The probability (frequency) of each person in the sample is 1/n.

32

Extension to continuous case: uniform distribution



$$E(X) = \int_0^1 x(1)dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

33

Symbol Interlude

- E(X) = μ
  - these symbols are used interchangeably
- Expected value is an extremely useful concept for good decision-making!

34

Example: the lottery

- The Lottery (also known as a tax on people who are bad at math…)
- A certain lottery works by picking 6 numbers from 1 to 49. It costs $1.00 to play the lottery, and if you win, you win $2 million after taxes.
- *If you play the lottery once, what are your expected winnings or losses?*

35

Lottery

Calculate the probability of winning in 1 try:

$$\frac{1}{\binom{49}{6}} = \frac{1}{\frac{49!}{43!6!}} = \frac{1}{13{,}983{,}816} = 7.2 \text{ x } 10^{-8}$$

"49 choose 6"

Out of 49 numbers, this is the number of distinct combinations of 6.

The probability function (note, sums to 1.0):

x$	*p(x)*
-1	**.999999928**
+ 2 million	***7.2 x 10^{-8}***

36

Expected Value

The probability function

x$	*p(x)*
-1	**.999999928**
+ 2 million	***7.2 x 10*** $^{-8}$

Expected Value

E(X) = P(win)*$2,000,000 + P(lose)*-$1.00
= 2.0 *x* 10^6 * 7.2 x 10^{-8}+ .999999928 (-1) = .144 - .999999928 = -$.86*

Negative expected value is never good!
You shouldn't play if you expect to lose money!

37

Expected Value

If you play the lottery every week for 10 years, what are your expected winnings or losses?

520 x (-.86) = -$447.20

38

Gambling (or how casinos can afford to give so many free drinks…)

A roulette wheel has the numbers 1 through 36, as well as 0 and 00. If you bet \$1 that an odd number comes up, you win or lose \$1 according to whether or not that event occurs. If random variable X denotes your net gain, X=1 with probability 18/38 and X= -1 with probability 20/38.

$E(X) = 1(18/38) - 1\,(20/38) = -\$.053$

On average, the casino wins (and the player loses) 5 cents per game.

The casino rakes in even more if the stakes are higher:

$E(X) = 10(18/38) - 10\,(20/38) = -\$.53$

If the cost is \$10 per game, the casino wins an average of 53 cents per game. If 10,000 games are played in a night, that's a cool \$5300.

39

**A few notes about Expected Value as a mathematical operator:

If c= a constant number (i.e., not a variable) and *X* and *Y* are any random variables…

- $E(c) = c$
- $E(cX)=cE(X)$
- $E(c + X)=c + E(X)$
- $E(X+Y)= E(X) + E(Y)$

40

E(c) = c

E(c) = c

Example: If you cash in soda cans in CA, you always get 5 cents per can.

Therefore, there's no randomness. You always expect to (and do) get 5 cents.

41

E(c*X*)=cE(*X*)

E(c*X*)=cE(*X*)

Example: If the casino charges $10 per game instead of $1, then the casino expects to make 10 times as much on average from the game (See roulette example above!)

42

$E(c + X) = c + E(X)$

$E(c + X) = c + E(X)$

Example, if the casino throws in a free drink worth exactly $5.00 every time you play a game, you always expect to (and do) gain an extra $5.00 regardless of the outcome of the game.

43

$E(X+Y) = E(X) + E(Y)$

$E(X+Y) = E(X) + E(Y)$

Example: If you play the lottery twice, you expect to lose: -$.86 + -$.86.

NOTE: This works even if X and Y are dependent!! Does not require independence!! Proof left for later…

44

Practice Problem

If a disease is fairly rare and the antibody test is fairly expensive, in a resource-poor region, one strategy is to take half of the serum from each sample and pool it with *n* other halved samples, and test the pooled lot. If the pooled lot is negative, this saves *n-1* tests. If it's positive, then you go back and test each sample individually, requiring *n+1* tests total.

a. Suppose a particular disease has a prevalence of 10% in a third-world population and you have 500 blood samples to screen. If you pool 20 samples at a time (25 lots), how many tests do you expect to have to run (assuming the test is perfect!)?
b. What if you pool only 10 samples at a time?
c. 5 samples at a time?

45

Answer (a)

a. Suppose a particular disease has a prevalence of 10% in a third-world population and you have 500 blood samples to screen. If you pool 20 samples at a time (25 lots), how many tests do you expect to have to run (assuming the test is perfect!)?

Let X = a random variable that is the number of tests you have to run per lot:

E(X) = P(pooled lot is negative)(1) + P(pooled lot is positive) (21)

$E(X) = (.90)^{20} (1) + [1-.90^{20}] (21) = 12.2\% (1) + 87.8\% (21) = 18.56$

E(total number of tests) = 25*18.56 = 464

46

Answer (b)

b. What if you pool only 10 samples at a time?

$E(X) = (.90)^{10}\,(1) + [1-.90^{10}]\,(11) = 35\%\,(1) + 65\%\,(11) = 7.5$ average per lot

50 lots * 7.5 = 375

47

Answer (c)

c. 5 samples at a time?

$E(X) = (.90)^{5}\,(1) + [1-.90^{5}]\,(6) = 59\%\,(1) + 41\%\,(6) = 3.05$ average per lot

100 lots * 3.05 = 305

48

Practice Problem

If X is a random integer between 1 and 10, what's the expected value of X?

49

Answer

If X is a random integer between 1 and 10, what's the expected value of X?

$$\mu = E(x) = \sum_{i=1}^{10} i\left(\frac{1}{10}\right) = \frac{1}{10}\sum_{i}^{10} i = (.1)\frac{10(10+1)}{2} = 55(.1) = 5.5$$

50

Expected value isn't everything though…

- Take the show "Deal or No Deal"
- Everyone know the rules?
- Let's say you are down to two cases left. $1 and $400,000. The banker offers you $200,000.
- So, Deal or No Deal?

51

Deal or No Deal…

- This could really be represented as a probability distribution and a non-random variable:

x$	*p(x)*
+1	**.50**
+$400,000	**.50**

x$	*p(x)*
+$200,000	**1.0**

52

Expected value doesn't help…

x$	*p(x)*
+1	.50
+$400,000	.50

$$\mu = E(X) = \sum_{\text{all } x} x_i p(x_i) = +1(.50) + 400{,}000(.50) = 200{,}000$$

x$	*p(x)*
+$200,000	1.0

$$\mu = E(X) = 200{,}000$$

53

How to decide?

Variance!

- **If you take the deal, the variance/standard deviation is 0.**
- **If you don't take the deal, what is average deviation from the mean?**
- **What's your gut guess?**

54

Variance/standard deviation

"The average (expected) squared distance (or deviation) from the mean"

$$\sigma^2 = Var(x) = E[(x-\mu)^2] = \sum_{\text{all } x} (x_i - \mu)^2 p(x_i)$$

***We square because squaring has better properties than absolute value. Take square root to get back linear average distance from the mean (="standard deviation").*

55

Variance, formally

Discrete case:

$$Var(X) = \sigma^2 = \sum_{\text{all } x} (x_i - \mu)^2 p(x_i)$$

Continuous case:

$$Var(X) = \sigma^2 = \int_{-\infty}^{\infty} (x_i - \mu)^2 p(x_i) dx$$

56

Similarity to empirical variance

The variance of a sample: s^2 =

$$\frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{n-1} = \sum_{i=1}^{N}(x_i - \bar{x})^2 \left(\frac{1}{n-1}\right)$$

Division by n-1 reflects the fact that we have lost a "degree of freedom" (piece of information) because we had to estimate the sample mean before we could estimate the sample variance.

57

Symbol Interlude

- Var(X) = σ^2
  - these symbols are used interchangeably

58

Variance: Deal or No Deal

$$\sigma^2 = \sum_{\text{all x}} (x_i - \mu)^2 p(x_i)$$

$$\sigma^2 = \sum_{\text{all x}} (x_i - \mu)^2 p(x_i) =$$

$$= (1 - 200{,}000)^2 (.5) + (400{,}000 - 200{,}000)^2 (.5) = 200{,}000^2$$

$$\sigma = \sqrt{200{,}000^2} = 200{,}000$$

Now you examine your personal risk tolerance…

59

Practice Problem

A roulette wheel has the numbers 1 through 36, as well as 0 and 00. If you bet $1.00 that an odd number comes up, you win or lose $1.00 according to whether or not that event occurs. If X denotes your net gain, X=1 with probability 18/38 and X= -1 with probability 20/38.

We already calculated the mean to be = -$.053. What's the variance of X?

60

Answer

$$\sigma^2 = \sum_{\text{all x}} (x_i - \mu)^2 p(x_i)$$

$$= (+1 - -.053)^2 (18/38) + (-1 - -.053)^2 (20/38)$$

$$= (1.053)^2 (18/38) + (-1 + .053)^2 (20/38)$$

$$= (1.053)^2 (18/38) + (-.947)^2 (20/38)$$

$$= .997$$

$$\sigma = \sqrt{.997} = .99$$

Standard deviation is $.99. Interpretation: On average, you're either 1 dollar above or 1 dollar below the mean, which is just under zero. Makes sense!

61

Handy calculation formula!

Handy calculation formula (if you ever need to calculate by hand!):

$$Var(X) = \sum_{\text{all x}} (x_i - \mu)^2 p(x_i) = \sum_{\text{all x}} x_i^{\,2} p(x_i) - (\mu)^2$$

Intervening algebra!

$$= E(x^2) - [E(x)]^2$$

62

$Var(x) = E(x-\mu)^2 = E(x^2) - [E(x)]^2$ (your calculation formula!)

Proofs (optional!):

$E(x-\mu)^2 = E(x^2 - 2\mu x + \mu^2)$ — remember "FOIL"?!

$= E(x^2) - E(2\mu x) + E(\mu^2)$ — Use rules of expected value: $E(X+Y) = E(X) + E(Y)$

$= E(x^2) - 2\mu E(x) + \mu^2$ — $E(c) = c$

$= E(x^2) - 2\mu\mu + \mu^2$ — $E(x) = \mu$

$= E(x^2) - \mu^2$

$= E(x^2) - [E(x)]^2$

OR, equivalently:

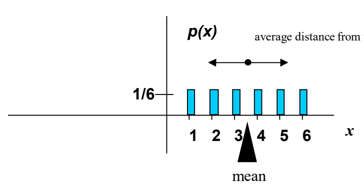
$E(x-\mu)^2 =$

$$\sum_{allx}[(x-\mu)^2]p(x) = \sum_{allx}[(x^2 - 2\mu x + \mu^2]p(x) = \sum_{allx} x^2 p(x) - 2\mu \sum xp(x) + \mu^2 \sum p(x) = E(x^2) - 2\mu E(x) + \mu^2(1) =$$
$$E(x^2) - 2\mu^2 + \mu^2(1) = E(x^2) - \mu^2$$

63

For example, what's the variance and standard deviation of the roll of a die?

x	p(x)
1	p(x=1)=1/6
2	p(x=2)=1/6
3	p(x=3)=1/6
4	p(x=4)=1/6
5	p(x=5)=1/6
6	p(x=6)=1/6
	1.0



$$E(x) = \sum_{\text{all } x} x_i p(x_i) = (1)(\frac{1}{6}) + 2(\frac{1}{6}) + 3(\frac{1}{6}) + 4(\frac{1}{6}) + 5(\frac{1}{6}) + 6(\frac{1}{6}) = \frac{21}{6} = 3.5$$

$$E(x^2) = \sum_{\text{all } x} x_i^2 p(x_i) = (1)(\frac{1}{6}) + 4(\frac{1}{6}) + 9(\frac{1}{6}) + 16(\frac{1}{6}) + 25(\frac{1}{6}) + 36(\frac{1}{6}) = 15.17$$

$$\sigma_x^2 = Var(x) = E(x^2) - [E(x)]^2 = 15.17 - 3.5^2 = 2.92$$
$$\sigma_x = \sqrt{2.92} = 1.71$$

64

**A few notes about Variance as a mathematical operator:

If c= a constant number (i.e., not a variable) and *X* and *Y* are random variables, then

- Var(c) = 0
- Var (c+*X*)= Var(*X*)
- Var(c*X*)= c^2Var*(X)*
- Var(*X*+*Y*)= Var(*X*) + Var(*Y*) ***ONLY IF X and Y are independent!!!!***
- {Var(*X*+*Y*)= Var(*X*) + Var(*Y*)+2Cov(X,Y) ***IF X and Y are not independent}***

65

Var(c) = 0

Var(c) = 0

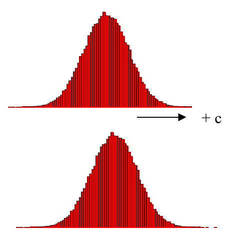
Constants don't vary!

66

Var (c+X)= Var(X)

Var (c+X)= Var(X)

Adding a constant to every instance of a random variable doesn't change the variability. It just shifts the whole distribution by c. If everybody grew 5 inches suddenly, the variability in the population would still be the same.

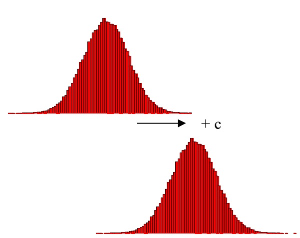


67

Var (c+X)= Var(X)

Var (c+X)= Var(X)

Adding a constant to every instance of a random variable doesn't change the variability. It just shifts the whole distribution by c. If everybody grew 5 inches suddenly, the variability in the population would still be the same.

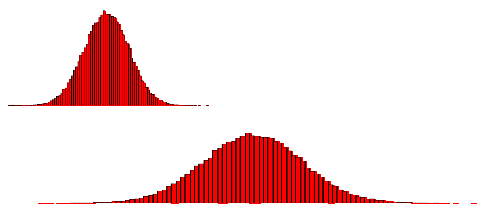


68

Var(cX)= c^2Var(X)

Var(cX)= c^2Var(X)

Multiplying each instance of the random variable by c makes it c-times as wide of a distribution, which corresponds to c^2 as much variance (deviation squared). For example, if everyone suddenly became twice as tall, there'd be twice the deviation and 4 times the variance in heights in the population.



69

Var(X+Y)= Var(X) + Var(Y)

Var(X+Y)= Var(X) + Var(Y) ***ONLY IF X and Y are independent!!!!!!!!***

With two random variables, you have more opportunity for variation, unless they vary together (are dependent, or have covariance): Var(X+Y)= Var(X) + Var(Y) + 2Cov(X, Y)

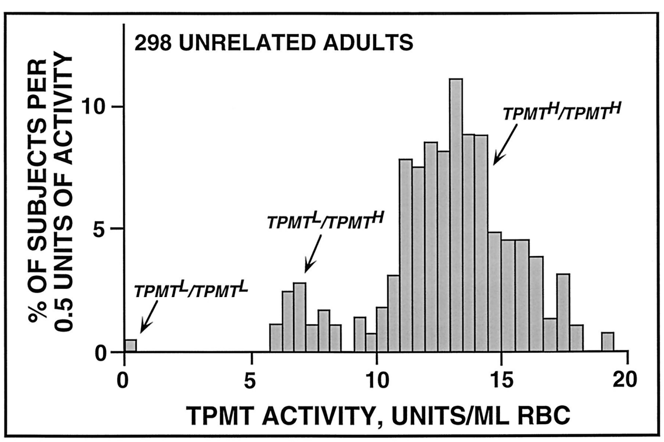
70

Example of Var(X+Y)= Var(X) + Var(Y): TPMT

- TPMT metabolizes the drugs 6-mercaptopurine, azathioprine, and 6-thioguanine (chemotherapy drugs)
- People with $TPMT^{-/}$ $TPMT^{+}$ have reduced levels of activity (10% prevalence)
- People with $TPMT^{-/}$ $TPMT^{-}$ have no TPMT activity (prevalence 0.3%).
- They cannot metabolize 6-mercaptopurine, azathioprine, and 6-thioguanine, and risk bone marrow toxicity if given these drugs.

71

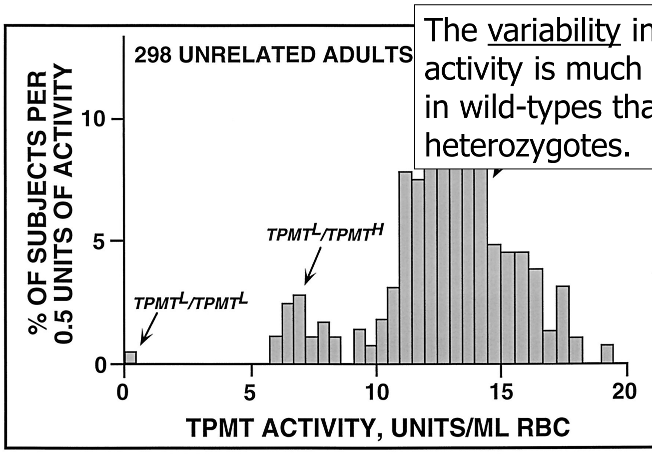
TPMT activity by genotype



Weinshilboum R. Drug Metab Dispos. 2001 Apr;29(4 Pt 2):601-5

72

TPMT activity by genotype

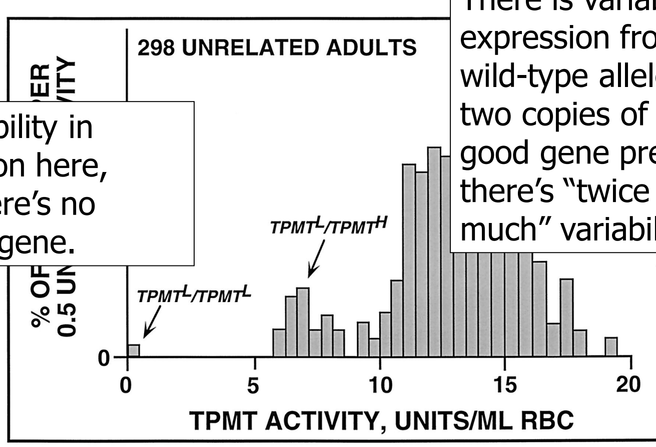


The variability in TPMT activity is much higher in wild-types than heterozygotes.

Weinshilboum R. Drug Metab Dispos. 2001 Apr;29(4 Pt 2):601-5

73

TPMT activity by genotype



No variability in expression here, since there's no working gene.

There is variability in expression from each wild-type allele. With two copies of the good gene present, there's "twice as much" variability.

Weinshilboum R. Drug Metab Dispos. 2001 Apr;29(4 Pt 2):601-5

74

Practice Problem

Find the variance and standard deviation for the number of ships to arrive at the harbor (recall that the mean is 11.3).

x	10	11	12	13	14
P(x)	.4	.2	.2	.1	.1

75

Answer: variance and std dev

x^2	100	121	144	169	196
P(x)	.4	.2	.2	.1	.1

$$E(x^2) = \sum_{i=1}^{5} x_i^2 p(x_i) = (100)(.4) + (121)(.2) + 144(.2) + 169(.1) + 196(.1) = 129.5$$

$$Var(x) = E(x^2) - [E(x)]^2 = 129.5 - 11.3^2 = 1.81$$

$$stddev(x) = \sqrt{1.81} = 1.35$$

Interpretation: On an average day, we expect 11.3 ships to arrive in the harbor, plus or minus 1.35. This gives you a feel for what would be considered a usual day!

76

Practice Problem

You toss a coin 100 times. What's the expected number of heads? What's the variance of the number of heads?

77

Answer: expected value

Intuitively, we'd probably all agree that we expect around 50 heads, right?

Another way to show this→
Think of tossing 1 coin. E(X=number of heads) = (1) P(heads) + (0)P(tails)

∴E(X=number of heads) = 1(.5) + 0 = .5
If we do this 100 times, we're looking for the sum of 100 tosses, where we assign 1 for a heads and 0 for a tails. (these are 100 "independent, identically distributed (i.i.d)" events)

$E(X_1 + X_2 + X_3 + X_4 + X_5 \ldots + X_{100}) = E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5) \ldots + E(X_{100}) =$
$100\, E(X_1) = 50$

78

Answer: variance

What's the variability, though? More tricky. But, again, we could do this for 1 coin and then use our rules of variance.

Think of tossing 1 coin.
$E(X^2$=number of heads squared$) = 1^2 P(\text{heads}) + 0^2 P(\text{tails})$

$\therefore E(X^2) = 1(.5) + 0 = .5$
$Var(X) = .5 - .5^2 = .5 - .25 = .25$
Then, using our rule: $Var(X+Y)= Var(X) + Var(Y)$ (coin tosses are independent!)
$Var(X_1 + X_2 + X_3 + X_4 + X_5 \ldots + X_{100}) = Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4) + Var(X_5) \ldots + Var(X_{100}) =$

$100\ Var(X_1) = 100\ (.25) = 25$
$SD(X)=5$

Interpretation: When we toss a coin 100 times, we expect to get 50 heads plus or minus 5.

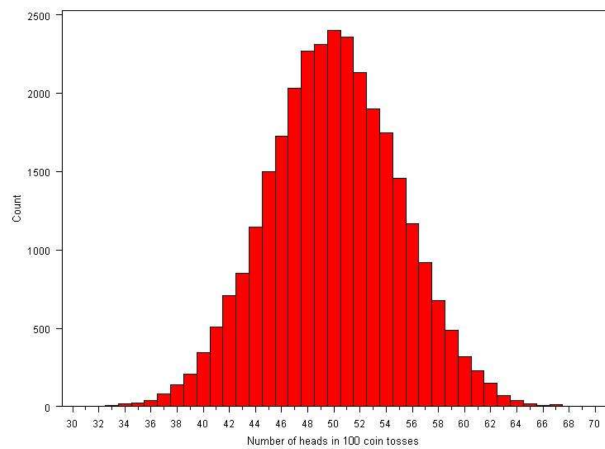
79

Or use computer simulation…

- Flip coins virtually!
  - Flip a virtual coin 100 times; count the number of heads.
  - Repeat this over and over again a large number of times (we'll try 30,000 repeats!)
  - Plot the 30,000 results.

80

Coin tosses…



Mean = 50

Std. dev = 5

Follows a normal distribution

∴95% of the time, we get between 40 and 60 heads…

81

Covariance: joint probability

- The covariance measures the strength of the linear relationship between two variables
- The covariance: $E[(x-\mu_x)(y-\mu_y)]$

$$\sigma_{xy} = \sum_{i=1}^{N} (x_i - \mu_x)(y_i - \mu_y)\, P(x_i, y_i)$$

82

The Sample Covariance

- The sample covariance:

$$\mathrm{cov}(x,y) = \frac{\sum_{i=1}^{n}(x_i - \overline{X})(y_i - \overline{Y})}{n-1}$$

83

Interpreting Covariance

- **Covariance** between two random variables:

cov(X,Y) > 0 ⟶ X and Y are positively correlated

cov(X,Y) < 0 ⟶ X and Y are inversely correlated

cov(X,Y) = 0 ⟶ X and Y are independent

84

Covariance example

	S&P 500	ABC Corp.
2013	1,692	68
2014	1,978	102
2015	1,884	110
2016	2,151	112
2017	2,519	154

$$\text{Mean (S\&P 500)} = \frac{1,692 + 1,978 + 1,884 + 2,151 + 2,519}{5} = 2,044.80$$

$$\text{Mean (ABC Corp.)} = \frac{68 + 102 + 110 + 112 + 154}{5} = 109.20$$

85

Covariance example

Step 3 (a, b); Step 4 (a x b)

	S&P 500	ABC Corp.	a	b	a x b
2013	1,692	68	-352.80	-41.20	14,535.36
2014	1,978	102	-66.80	-7.20	480.96
2015	1,884	110	-160.80	0.80	-128.64
2016	2,151	112	106.20	2.80	297.36
2017	2,519	154	474.20	44.80	21,244.16
Mean	**2,044.80**	**109.20**	**Sum**		**36,429.20**

$$\text{Cov(S\&P 500, ABC Corp.)} = \frac{36,429.20}{5 - 1} = 9,107.30$$

86